OLIMPIADA NAȚIONALĂ DE MATEMATICĂ
ETAPA LOCALĂ 19.02.2015

barem clasa a XI-a matematică-informatică

1. a) $\det\left(A^2+B^2\right)=\det\left(A+iB\right)\left(A-iB\right)=$ 1p

$=\det\left(A+iB\right)\cdot\det\overline{\left(A+iB\right)}=$ 1p

$=\det\left(A+iB\right)\cdot\overline{\det\left(A+iB\right)}=\left|\det\left(A+iB\right)\right|^2\geq 0$ 1p

b) $\det\left(A^2-A+I_n\right)=\det\left(\left(A-\frac{1}{2}I_n\right)^2+\left(\frac{\sqrt{3}}{2}I_n\right)^2\right)\geq 0$ 2p

$\det\left(A^2-A+I_n\right)=\det A\left(A-I_n+A^{-1}\right)=\det A\cdot\det A\left(A-I_n+A^{-1}\right)$ 1p

$\det A>0\Rightarrow\det\left(A-I_n+A^{-1}\right)\geq 0$ 1p

2. Din $\left(A+B\right)^2=A^2+B^2\Rightarrow AB+BA=O_n$ 1p

Din $A\left(AB+BA\right)B=O_n\Rightarrow A^2B^2+\left(AB\right)^2=O_n$ 1p

Din $B\left(AB+BA\right)A=O_n\Rightarrow\left(BA\right)^2+B^2A^2=O_n$ 1p

Din $\left(A+B\right)^4=A^4+B^4$ şi $\left(A+B\right)^4=\left(A+B\right)^2\left(A+B\right)^2$ rezultă $A^2B^2+B^2A^2=O_n$ 1p

Se obține $\left(AB\right)^2+\left(BA\right)^2=O_n$ 1p

Din $AB+BA=O_n\Rightarrow AB=-BA$ 1p

Atunci $\left(AB\right)^2+\left(-AB\right)^2=O_n\Rightarrow\left(AB\right)^2=O_n$ 1p

3. a) Din $a_{n+1}-a_n=\ln\left(1+n+n^2\right)\geq\ln 1=0,\ \forall n\geq 1$ rezultă că şirul $\left(a_n\right)_{n\geq 1}$ este crescător..1p

Dacă şirul ar fi mărginit, ar fi convergent cu $\lim\limits_{n\to\infty}a_n=l\in\mathbb{R}$ şi din relația de recurență ar rezulta $l=l+\infty$, absurd, deci şirul este divergent 1p

b) Din a) $\lim\limits_{n\to\infty}a_n=\infty$ şi din $b_n=n\cdot\ln n$ strict crescător şi nemărginit 1p

se calculează $\lim\limits_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}=\lim\limits_{n\to\infty}\frac{\ln\left(1+n+n^2\right)}{\left(n+1\right)\ln\left(n+1\right)-n\ln n}=$ 1p

$=\lim\limits_{n\to\infty}\frac{\ln\left(1+n+n^2\right)}{\ln\frac{\left(n+1\right)^{n+1}}{n^n}}=\lim\limits_{n\to\infty}\frac{\ln\left(1+n+n^2\right)}{\ln\left(1+\frac{1}{n}\right)^n+\ln\left(n+1\right)}=$ 1p

$$=\lim_{n\to\infty}\frac{\ln\left(1+n+n^2\right)}{\ln\left(n+1\right)\left(1+\frac{\ln\left(1+\frac{1}{n}\right)^n}{\ln\left(n+1\right)}\right)}=\lim_{n\to\infty}\frac{\ln\left(n^2\left(\frac{1}{n^2}+\frac{1}{n}+1\right)\right)}{\ln\left(n\left(\frac{1}{n}+1\right)\right)}=\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{1p}$$

$$=\lim_{n\to\infty}\frac{2\ln n+\ln\left(\frac{1}{n^2}+\frac{1}{n}+1\right)}{\ln n+\ln\left(\frac{1}{n}+1\right)}=\lim_{n\to\infty}\frac{\ln n\left(2+\frac{\ln\left(\frac{1}{n^2}+\frac{1}{n}+1\right)}{\ln n}\right)}{\ln n\left(1+\frac{\ln\left(\frac{1}{n}+1\right)}{\ln n}\right)}=2\Rightarrow\lim_{n\to\infty}\frac{a_n}{n\cdot\ln n}=2 \text{ conform}$$

lemei lui Stolz – Cesaro..1p

4. a) Prin inducție matematică se arată că $a_n=n^2,\ \forall n\in\mathbb{N}^*$ 2p

$\lim\limits_{n\to\infty}a_n=\infty$ 1p

b) $$l=\lim_{n\to\infty}\left(n^2\left(\sum_{k=1}^{p}\ln\left(1+\cos\frac{k}{n}-1\right)\right)\right)=\lim_{n\to\infty}\sum_{k=1}^{p}\frac{\ln\left(1-2\sin^2\frac{k}{2n}\right)}{-2\sin^2\frac{k}{2n}}\cdot\frac{-2\sin^2\frac{k}{2n}}{\left(\frac{k}{2n}\right)^2}\cdot\frac{k^2}{4n^2}\cdot n^2=\ldots\text{2p}$$

$$=-\frac{1}{2}\sum_{k=1}^{p}k^2=-\frac{p\left(p+1\right)\left(2p+1\right)}{12}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{1p}$$

Se obține $p=3$ 1p